Proof applies equally well to the advanced potentials.

$$V_a(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r', t_a)}{\mathscr{r}}\, d\tau'$$

$$A_a(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r', t_a)}{\mathscr{r}}\, d\tau' \quad \text{— (9)}$$

Charge & current densities are evaluated at the advanced time

$$t_a \equiv t + \frac{\mathscr{r}}{c} \quad \text{— (10)}$$

⇒ Suggest that the potential now depend on what the charge and the current distribution will be at some time in future

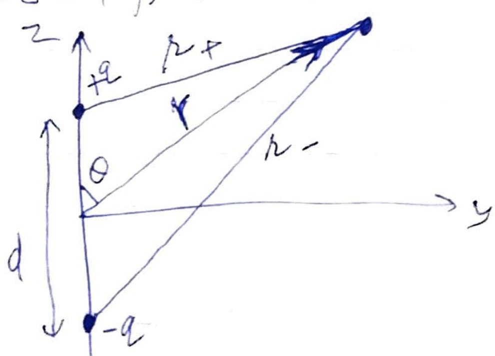
↳ effect, precedes the charge

Electric dipole Radiation

Two tiny metal spheres separated by a distance 'd' and connected by a fine wire

Charge on upper sphere $q(t)$

lower sphere $-q(t)$



We drive the charge back and forth through the wire, from one end to other at an angular frequency ω:

$$q(t) = q_0 \cos(\omega t) \quad \text{—— (1)}$$

Result is an oscillating electric dipole

$$p(t) = p_0 \cos(\omega t)\hat{z} \quad \text{—— (2)}$$

where $p_0 \equiv q_0 d$ ⟶ Maximum value of dipole moment.

The retarded potential is

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0\cos[\omega(t-r_+/c)]}{r_+} - \frac{q_0\cos[\omega(t-r_-/c)]}{r_-}\right]$$

by the law of cosines —— (3)

$$r_\pm = \sqrt{r^2 \mp r d\cos\theta + \left(\frac{d}{2}\right)^2}$$

physical dipole → perfect dipole —— (4)

separation distance to be extremely small

approximation 1 : $d << r$

$$r_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right) \quad \text{—— (5)}$$

[expansion in first order in d]

It follows that

$$\frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad \text{—— (6)}$$

and

$$\cos[\omega(t - r_\pm/c)] \cong \cos\left[\omega\left(t-\frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

In the perfect dipole limit, we have further

Approximation 2 : $d << \frac{c}{\omega}$

Since wave of frequency ω have a wavelength $\lambda = \frac{2\pi c}{\omega}$ this amounts to the requirement $d << \lambda$

$$\Rightarrow \cos\left[\omega(t - r_\pm/c)\right] \cong \cos\left[\omega\left(t-\frac{r}{c}\right)\right] \mp \frac{\omega d}{2c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right] \quad \text{—— (7)}$$

Putting values from eqn (6) & (7) into eqn (3)

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \quad \text{—(8)}$$

In the static limit ($\omega \to 0$) the second term reproduces the old formula for the potential of a stationary dipole

$$\boxed{V = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r^2}}$$

We are interested in the fields that survive at large distances from the source, i.e. in the so-called radiation zone $d \ll \lambda \ll r$

Approximation 3: $r \gg \frac{c}{\omega}$

or in terms of wavelength, $r \gg \lambda$

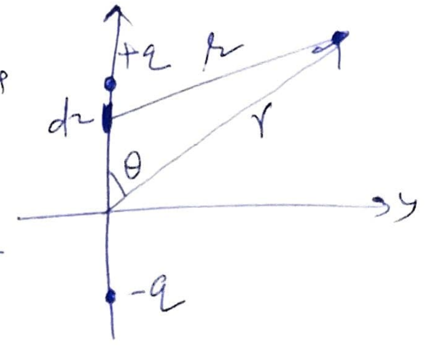
In this region the potential reduces to

$$\boxed{V(r,\theta,t) = -\frac{p_0 \omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \quad \text{—(9)}$$

The vector potential is determined by the current flowing in the wire:

$$I(t) = \frac{dq}{dt}\hat{z} = -q_0 \omega \sin(\omega t)\,\hat{z} \quad \text{—(10)}$$

from figure



$$A(r,t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0 \omega \sin\left[\omega\left(t-\frac{𝓇}{c}\right)\right]\hat{z}}{𝓇}\,dz \quad \text{—(11)}$$

To first order, we can replace the integrand by its value at the center

$$A(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z} \quad \text{—(12)}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$\nabla V \cong \frac{P_0\omega^2}{4\pi\varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\,\hat{r}$$

[use approximation 3]

$$\frac{\partial A}{\partial t} = -\frac{\mu_0 P_0\omega^2}{4\pi r}\cos[\omega(t-\frac{r}{c})](\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$E = -\nabla V - \frac{\partial A}{\partial t} = -\frac{\mu_0 P_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\,\hat{\theta} \quad (13)$$

$$\nabla \times A = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$$

$$= -\frac{\mu_0 P_0\omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-\frac{r}{c})] + \frac{\sin\theta}{r}\sin[\omega(t-\frac{r}{c})]\right\}\hat{\phi}$$

E & B monochromatic waves of frequency ω, travelling in radial direction at speed of light

Second term eliminated by approx. 3

Electromagnetic waves in free space

$$B = \nabla \times A = -\frac{\mu_0 P_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\,\hat{\phi} \quad (14)$$

E & B in phase mutually perpendicular and transverse $\frac{E_0}{B_0} = c$

The energy radiated by an oscillating electric dipole $\rightarrow$ Poynting vector

$$S = \frac{1}{\mu_0}(E \times B) = \frac{\mu_0}{c}\left\{\frac{P_0\omega^2}{4\pi}\left[\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\right]\right\}^2\hat{r} \quad (15)$$

Intensity is obtained by averaging (in time) over a complete cycle:

$$\langle S\rangle = \left(\frac{\mu_0 P_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{r} \quad (16)$$

No radiation along the axis of dipole
[here $\sin\theta = 0$]

The intensity profile takes the form of donut
The total power radiated is found by integrating $\langle S \rangle$ over a sphere of radius r:

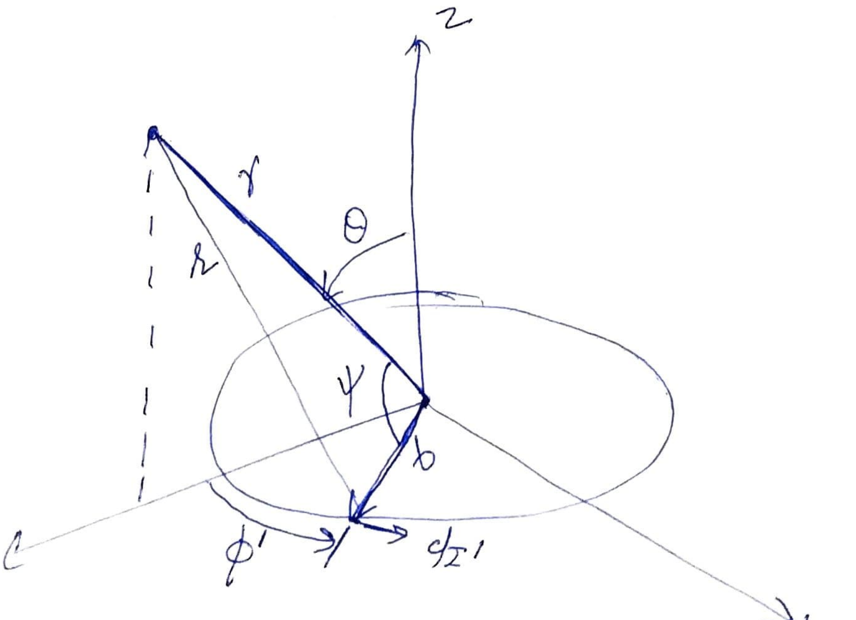
$$\langle P \rangle = \int \langle S \rangle \cdot da$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \quad \text{---} (17)$$

It is independent of the radius of the sphere

Magnetic Dipole Radiation



We have a wire loop of radius b, around which we can drive an alternating current:

$$I(t) = I_0 \cos\omega t \quad \text{---} (18)$$

$$m(t) = \pi b^2 I(t)\,\hat{z}$$
$$= m_0 \cos(\omega t)\,\hat{z} \quad \text{--- (19)}$$

where $m_0 \equiv \pi b^2 I_0$ --- (20)

is the maximum value of the magnetic dipole moment

The loop is uncharged so the scalar potential is zero. The retarded vector potential is

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{I_0 \cos[\omega(t - \frac{\imath}{c})]}{\imath}\, dI' \quad \text{--- (21).}$$

For a point r directly above x-axis.

$A \rightarrow$ aims in the y-direction. Since the x-component from symmetrically placed points on either side of the x-axis will cancel.

$$A(r,t) = \frac{\mu_0 I b}{4\pi}\,\hat{y}\int_0^{2\pi} \frac{\cos[\omega(t - \frac{\imath}{c})]}{\imath}\cos\phi'\, d\phi' \quad \text{--- (22)}$$

By the law of cosines

$$\imath = \sqrt{r^2 + b^2 - 2rb\cos\psi},$$

$\psi \rightarrow$ angle between the vectors r and b:

$$r = r\sin\theta\,\hat{x} + r\cos\theta\,\hat{z}$$
$$b = b\cos\phi'\,\hat{x} + b\sin\phi'\,\hat{y}$$

So $rb\cos\psi = r\cdot b = rb\sin\theta\cos\phi'$, and therefore

$$\imath = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

For a "perfect" dipole, we want the loop to be extremely small:

Approximation 1: $b \ll r$ -- (23)

To first order in b, then

$$\imath \cong r\left(1-\frac{b}{r}\sin\theta\cos\phi'\right),$$

so $$\frac{1}{\imath} \cong \frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right) \quad \text{— (24)}$$

and

$$\cos\left[\omega\left(t-\frac{\imath}{c}\right)\right] \cong \cos\left[\omega\left(t-\frac{r}{c}\right)+\frac{\omega b}{c}\sin\theta\cos\phi'\right]$$

$$= \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) - \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right).$$

Approximation 2: $b \ll \frac{c}{\omega}$ — (25)

Size of the dipole is small compared to the wavelength radiated:

$$\cos\left[\omega\left(t-\frac{\imath}{c}\right)\right] \cong \cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega b}{c}\sin\theta\cos\phi'\sin\left[\omega\left(t-\frac{r}{c}\right)\right] \quad \text{— (26)}$$

In sertting eqn (24) & (26) into eqn (22) and dropping the second-order term

$$A(r,t) \cong \frac{\mu_0 I_0 b}{4\pi r}\hat{y}\int_0^{2\pi}\Big\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\Big\}\cos\phi'\,d\phi'$$

The first term integrates to zero:

$$\int_0^{2\pi}\cos\phi'\,d\phi' = 0$$

The second term involves the integral of cosine squared:

$$\int_0^{2\pi}\cos^2\phi'\,d\phi' = \pi$$

Putting this in, and noting that in general A points in the $\hat{\phi}$-direction.

The vector potential of an oscillating perfect magnetic dipole is

$$A(r,\theta,t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t-\tfrac{r}{c})] - \frac{\omega}{c}\sin[\omega(t-\tfrac{r}{c})]\right\}\hat{\phi} \quad \text{--- (27)}$$

In the static limit $(\omega=0)$

$$\boxed{A(r,\theta) = \frac{\mu_0}{4\pi}\frac{m_0 \sin\theta}{r^2}\hat{\phi}}$$

In the radiation zone

Approximation 3: $r >> \frac{c}{\omega}$ --- (28)

first term in A is negligible, so

$$A(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t-\tfrac{r}{c})]\hat{\phi} \quad \text{--- (29)}$$

From A, we obtain fields at large r:

$$E = -\frac{\partial A}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\tfrac{r}{c})]\hat{\phi} \quad \text{--- (30)}$$

and

$$B = \nabla\times A = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\tfrac{r}{c})]\hat{\theta} \quad \text{--- (31)}$$

[using approx. 3]

fields are in phase, mutually perpendicular and transverse to the direction of propagation $(\hat{r})$ and ratio of their amplitudes E_0 ...

all of which as expected for an electromagnetic wave. Remarkably similar in structure to the fields of an oscillating electric dipole.

The energy flux for magnetic dipole radiation is

$$S = \frac{1}{\mu_0}(E \times B) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}^2 \hat{r} \quad —(32)$$

The intensity is

$$\langle S \rangle = \left(\frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\hat{r}, \quad —(33)$$

Total radiated power is

$$\langle P \rangle = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3} \quad —(34)$$

Comparing

$$\frac{P_{magnetic}}{P_{electric}} = \left(\frac{m_0}{p_0 c}\right)^2 \quad —(35)$$

$m_0 = \pi b^2 I_0$, $p_0 = q_0 d$

amplitude of current in the electric case $I_0 = q_0\omega$

setting $d = \pi b$, for sake of comparison

$$\frac{P_{magnetic}}{P_{electric}} = \left(\frac{\omega b}{c}\right)^2 \quad —(36)$$

approx. 2 $\frac{\omega b}{c} \to$ very small

Ordinarily one should expect electric dipole radiation to dominate.